

# COACTIONS AND SKEW PRODUCTS OF TOPOLOGICAL GRAPHS

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ABSTRACT. We show that the  $C^*$ -algebra of a skew-product topological graph  $E \times_\kappa G$  is isomorphic to the crossed product of  $C^*(E)$  by a coaction of the locally compact group  $G$ .

## 1. INTRODUCTION

In [4, Theorem 2.4] we proved that if  $E$  is a directed graph and  $\kappa$  is a function from the edges of  $E$  to a discrete group  $G$ , then the graph algebra  $C^*(E \times_\kappa G)$  of the skew-product graph is a crossed product of  $C^*(E)$  by a coaction of  $G$ . This was later generalized to homogeneous spaces  $G/H$  in [2, Theorem 3.4], and to higher-rank graphs in [9, Theorem 7.1]. In this paper we generalize the result to topological graphs and locally compact groups. More precisely, we prove in Theorem 3.1 that if  $\kappa: E \rightarrow G$  is a continuous function (that is, a *cocycle*), then there exists a coaction  $\varepsilon$  of  $G$  on  $C^*(E)$  such that

$$C^*(E \times_\kappa G) \cong C^*(E) \times_\varepsilon G.$$

We give two distinct approaches to the coaction: in Section 3 we obtain the coaction indirectly, via an application of Landstad duality, and in Section 4 we construct the coaction directly, applying techniques developed in [6]. We thank Iain Raeburn for helpful conversations concerning this direct approach.

In Section 2 we record our conventions for topological graphs,  $C^*$ -correspondences, skew products, multiplier modules, and functoriality of Cuntz-Pimsner algebras. In an appendix we develop a few tools that we need for dealing with certain bimodule multipliers in terms of function spaces.

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## 2. PRELIMINARIES

In general, we refer to [11] (see also [8]) for topological graphs, and to [11, 3] (see also [7]) for  $C^*$ -correspondences, except we make a few minor, self-explanatory modifications. Thus, a topological graph  $E$  comprises locally compact Hausdorff spaces  $E^1, E^0$  and maps  $s, r: E^1 \rightarrow E^0$  with  $s$  a local homeomorphism and  $r$  continuous. Let  $A = C_0(E^0)$ , and let  $X = X(E)$  be the associated  $A$ -correspondence, which is the completion of  $C_c(E^1)$  with operations defined for  $f \in A$  and  $\xi, \eta \in C_c(E^1)$  by

$$\begin{aligned} f \cdot \xi(e) &= f(r(e))\xi(e) \\ \xi \cdot f(e) &= \xi(e)f(s(e)) \\ \langle \xi, \eta \rangle(v) &= \sum_{s(e)=v} \overline{\xi(e)}\eta(e). \end{aligned}$$

Throughout this paper we will also write  $A' = C_0(E^1)$ , so that  $X$  can be regarded as an  $A' - A$  correspondence as well as an  $A$ -correspondence. Recall from [8] that the left  $A'$ -module multiplication is nondegenerate in the sense that  $A' \cdot X = X$ , and is determined by the homomorphism  $\pi_E: A' \rightarrow \mathcal{L}(X)$  given by  $(\pi_E(f)\xi)(e) = f(e)\xi(e)$  for  $f \in A'$  and  $\xi \in C_c(E^1)$ , and the (nondegenerate) left  $A$ -module multiplication  $\varphi_A: A \rightarrow \mathcal{L}(X)$  is then given by  $\varphi_A(f) = \pi_E(1 \circ r)$  for  $f \in A$ .

We denote by  $(k_X, k_A): (X, A) \rightarrow C^*(E) = \mathcal{O}_X$  the universal Cuntz-Pimsner covariant representation, and for any Cuntz-Pimsner covariant representation  $(\psi, \pi)$  of  $(X, A)$  in a  $C^*$ -algebra  $B$  we denote by  $\psi^{(1)}: \mathcal{K}(X) \rightarrow B$  the associated homomorphism<sup>2</sup> determined by  $\psi^{(1)}(\theta_{\xi, \eta}) = \psi(\xi)\psi(\eta)^*$ , and by  $\psi \times \pi: C^*(E) \rightarrow B$  the unique homomorphism satisfying

$$(\psi \times \pi) \circ k_X = \psi \quad \text{and} \quad (\psi \times \pi) \circ k_A = \pi.$$

Note that in [3], correspondences were called right-Hilbert bimodules, and nondegeneracy was built into the definition. All our correspondences will in fact be nondegenerate, so we can freely apply the results from [3].

For skew products of topological graphs, we use a slight variation of the definition in [1]: the main difference is that we use the same notational conventions as those in [11] for skew products of discrete directed graphs. Thus, a *cocycle* of a locally compact group  $G$  on a

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<sup>2</sup>and here we use the notation of [1]; Raeburn would write  $(\psi, \pi)^{(1)}$

topological graph  $E$  is a continuous map  $\kappa: E^1 \rightarrow G$ , and the *skew product* is the topological graph  $E \times_\kappa G$  with

$$(E \times_\kappa G)^i = E^i \times G \quad (i = 0, 1),$$

$$r(e, t) = (r(e), \kappa(e)t), \quad \text{and} \quad s(e, t) = (s(e), t).$$

Our conventions for multipliers of correspondences are taken primarily from [3, Chapter 1], but also see [7]. If  $(\pi, \psi, \tau): (A, X, B) \rightarrow (M(C), M(Y), M(D))$  is a correspondence homomorphism, then there is a unique homomorphism  $\psi^{(1)}: \mathcal{K}(X) \rightarrow M(\mathcal{K}(Y)) = \mathcal{L}(Y)$  such that  $\psi^{(1)}(\theta_{\xi, \eta}) = \psi(\xi)\psi(\eta)^*$  for  $\xi, \eta \in X$ . (For this result in the stated level of generality, in particular with no nondegeneracy assumption on  $(\psi, \pi)$ , see [7, Lemma 2.1].) If  $(\psi, \pi)$  happens to be nondegenerate, then so is  $\psi^{(1)}$ , and hence  $\psi^{(1)}$  extends uniquely to a homomorphism  $\overline{\psi^{(1)}}: \mathcal{L}(X) \rightarrow \mathcal{L}(Y)$ .

A correspondence homomorphism  $(\psi, \pi): (X, A) \rightarrow (M(Y), M(B))$  is defined in [7] to be *Cuntz-Pimsner covariant* if

- (i)  $\psi(X) \subset M_B(Y)$ ,
- (ii)  $\pi: A \rightarrow M(B)$  is nondegenerate,
- (iii)  $\pi(J_X) \subset M(B; J_Y)$ , and
- (iv) the diagram

$$\begin{array}{ccc} J_X & \xrightarrow{\pi|} & M(B; J_Y) \\ \varphi_A| \downarrow & & \downarrow \overline{\varphi_B}| \\ \mathcal{K}(X) & \xrightarrow{\psi^{(1)}} & M_B(\mathcal{K}(Y)) \end{array}$$

commutes,

where, for an ideal  $I$  of a  $C^*$ -algebra  $C$ ,

$$M(C; I) := \{m \in M(C) : mC \cup Cm \subset I\}.$$

By [7, Corollary 3.6], for each Cuntz-Pimsner covariant homomorphism  $(\psi, \pi)$ , there is a unique homomorphism  $\mathcal{O}_{\psi, \pi}$  making the diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{(\psi, \pi)} & (M_B(Y), M(B)) \\ (k_X, k_A) \downarrow & & \downarrow (\overline{k_Y}, \overline{k_B}) \\ \mathcal{O}_X & \xrightarrow{\mathcal{O}_{\psi, \pi}} & M_B(\mathcal{O}_Y) \end{array}$$

commute. Moreover,  $\mathcal{O}_{\psi, \pi}$  is nondegenerate, and is injective if  $\pi$  is.

Our conventions for coactions on correspondences mainly follow [3], but see also [6].

## 3. INDIRECT APPROACH

In this section we apply Landstad duality to give an indirect approach to the following result:

**Theorem 3.1.** *If  $\kappa: E^1 \rightarrow G$  is a cocycle on a topological graph  $E$ , then there is a coaction  $\varepsilon$  of  $G$  on  $C^*(E)$  such that*

$$C^*(E \times_\kappa G) \cong C^*(E) \times_\varepsilon G.$$

Throughout the rest of this paper, in addition to  $A = C_0(E^0)$  and  $X = X(E)$ , we will also use the following abbreviations:

- $F = E \times_\kappa G$ ;
- $Y = X(E \times_\kappa G)$ ;
- $B = C_0((E \times_\kappa G)^0)$ .

*Proof.* To apply Landstad duality [10, Theorem 3.3] (stated in more modern form in [5, Theorem 4.1]), we need the following ingredients: an action  $\alpha: G \rightarrow \text{Aut } C^*(F)$ , a  $\text{rt}-\alpha$  equivariant nondegenerate homomorphism  $\mu: C_0(G) \rightarrow M(C^*(F))$  (where “rt” is action of  $G$  on  $C_0(G)$  by right translation), and an injective nondegenerate homomorphism

$$\Pi: C^*(E) \rightarrow M(C^*(F))$$

whose image coincides with Rieffel’s generalized fixed-point algebra  $C^*(F)^\alpha$ . Note that in [5],  $C^*(F)^\alpha$  would be written as  $\text{Fix}(C^*(F), \alpha, \mu)$ .

Since  $G$  acts on the right of the skew-product topological graph  $F$  via right translation in the second coordinate, by [1, Proposition 5.4 and discussion preceding Remark 5.3] we have an action  $\beta = (\beta^1, \beta^0): G \rightarrow \text{Aut } Y$  such that

$$\begin{aligned} \beta_t^1(\xi)(e, r) &= \xi(e, rt) \quad \text{for } \xi \in Y \\ \beta_t^0(g)(v, r) &= g(v, rt) \quad \text{for } g \in B, \end{aligned}$$

which in turn gives an action on  $C^*(F)$  such that

$$\begin{aligned} \alpha_t \circ k_Y &= k_Y \circ \beta_t^1 \\ \alpha_t \circ k_B &= k_B \circ \beta_t^0. \end{aligned}$$

Since  $F^0 = E^0 \times G$ , we have

$$B = C_0(F^0) = C_0(E^0) \otimes C_0(G) = A \otimes C_0(G),$$

so we can define a nondegenerate homomorphism  $\mu: C_0(G) \rightarrow M(C^*(F))$  by

$$\mu(g) = \overline{k_B}(1_{M(A)} \otimes g),$$

and then it is routine to verify that  $\mu$  is  $\text{rt} - \alpha$  equivariant.

Finally, since the action of  $G$  on  $F$  is free and proper, the proof of [1, Theorem 5.6] constructs an isomorphism

$$\Pi: C^*(E) \xrightarrow{\cong} C^*(F)^\alpha,$$

and then the result follows from Landstad duality.  $\square$

#### 4. A DIRECT APPROACH TO THE COACTION

As in Section 3, we suppose we are given a cocycle  $\kappa: E^1 \rightarrow G$  of a locally compact group  $G$  on a topological graph  $E$ , and we continue to write  $A = C_0(E^0)$ ,  $A' = C_0(E^1)$ ,  $X = X(E)$ ,  $F = E \times_\kappa G$ ,  $Y = X(F)$ , and  $B = C_0(F^0)$ .

Recall that the canonical embedding  $G \hookrightarrow M(C^*(G))$  is identified with a unitary element  $w_G$  of  $M(C_0(G) \otimes C^*(G))$ . Similarly, we may identify  $\kappa$  with a unitary element of

$$C_b(E^1, M^\beta(C^*(G))) = M(A' \otimes C^*(G)),$$

where  $M^\beta(C^*(G))$  denotes the multiplier algebra  $M(C^*(G))$  with the strict topology.

Define a nondegenerate homomorphism  $\kappa^*: C_0(G) \rightarrow M(A')$  by  $\kappa^*(f) = f \circ \kappa$ , and a nondegenerate homomorphism  $\nu: C_0(G) \rightarrow \mathcal{L}(X)$  by

$$\nu = \pi_E \circ \kappa^*,$$

where  $\pi_E: A' \rightarrow \mathcal{L}(X)$  is the homomorphism given on  $C_c(E^1)$  by point-wise multiplication.

**Proposition 4.1.** *With the above notation, there is a coaction  $(\sigma, \text{id}_A \otimes 1)$  of  $G$  on  $(X, A)$  defined by*

$$\sigma(\xi) = v \cdot (\xi \otimes 1),$$

where

$$v = \overline{\nu \otimes \text{id}}(w_G) \in \mathcal{L}(X \otimes C^*(G)),$$

and moreover there is a coaction  $\zeta$  of  $G$  on  $C^*(E)$  such that

$$\begin{aligned} \zeta \circ k_X &= \overline{k_X \otimes \text{id}} \circ \sigma \\ \zeta \circ k_A &= k_A \otimes 1. \end{aligned}$$

*Proof.* This follows from [6, Corollaries 3.4–3.5], because  $\nu: C_0(G) \rightarrow \mathcal{L}(X)$  commutes with  $\varphi_A$ .  $\square$

It will be convenient for us to find an equivalent expression for the coaction  $\sigma$ . Note that we may regard  $X$  as an  $A' - A$  correspondence,

and hence  $X \otimes C^*(G)$  as an  $(A' \otimes C^*(G)) - (A \otimes C^*(G))$  correspondence. Thus we can write

$$\sigma(\xi) = \overline{\kappa^* \otimes \text{id}}(w_G) \cdot (\xi \otimes 1).$$

However, we can go further: by construction the unitary element  $\overline{\kappa^* \otimes \text{id}}(w_G)$  of  $M(A' \otimes C^*(G))$  coincides with the function in  $C_b(E^1, M^\beta(C^*(G)))$  whose value at an edge  $e$  is

$$\overline{\kappa^* \otimes \text{id}}(w_G)(e) = w_G(\kappa(e)) = \kappa(e);$$

thus we can write

$$\sigma(\xi) = \kappa \cdot (\xi \otimes 1).$$

In Theorem 3.1 we used Landstad duality to show that  $C^*(F)$  is isomorphic to the crossed product of  $C^*(E)$  by a coaction  $\varepsilon$  of  $G$ ; on the other hand, in Proposition 4.1 we directly constructed a coaction  $\zeta$  of  $G$  on  $C^*(E)$ . To show that also  $C^*(E) \times_\zeta G \cong C^*(F)$ , we now show that in fact the coactions  $\varepsilon$  and  $\zeta$  coincide. Since the mechanism behind Landstad duality is that  $\varepsilon$  is pulled back along  $\Pi^{-1}$  from the inner coaction  $\delta^\mu$  on  $C^*(F)$ , this is accomplished by the following:

**Proposition 4.2.** *Let  $\zeta$  be the coaction on  $C^*(E)$  from Proposition 4.1, and let  $\Pi: C^*(E) \rightarrow M(C^*(F))$  and  $\mu: C_0(G) \rightarrow M(C^*(F))$  be as in the proof of Theorem 3.1. Then  $\Pi$  is  $\zeta - \delta^\mu$  equivariant, and hence  $\zeta$  coincides with the coaction  $\varepsilon$  from Theorem 3.1.*

*Proof.* It is equivalent to show that  $(\Pi, \mu): (C^*(E), C_0(G)) \rightarrow M(C^*(F))$  is a covariant representation for the coaction  $\zeta$ , and for this we will apply [6, Corollary 4.3].

We will need to know how the homomorphism  $\Pi$  from [1] can be described using the techniques of [7]: [1, Proof of Theorem 5.6] constructs a correspondence homomorphism

$$(\psi, \pi): (X, A) \rightarrow (M_B(Y), M(B)),$$

although the notation in [1] is substantially different<sup>3</sup>. In the terminology of [7, Definition 3.1], [1, Proof of Theorem 5.6] shows that  $(\psi, \pi)$  is Cuntz-Pimsner covariant, so that by [7, Corollary 3.6] there is a nondegenerate homomorphism  $\mathcal{O}_{\psi, \pi}$  making the diagram

$$\begin{array}{ccc} (X, A) & \xrightarrow{(\psi, \pi)} & (M_B(Y), M(B)) \\ \downarrow (k_X, k_A) & & \downarrow (\overline{k_Y}, \overline{k_B}) \\ C^*(E) & \xrightarrow{\mathcal{O}_{\psi, \pi}} & M_B(C^*(F)) \end{array}$$

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<sup>3</sup>The roles of  $E, X, A$  and  $F, Y, B$  are interchanged, and what we call  $(\psi, \pi)$  here was written as  $(\mu, \nu)$  in [1].

commute; the homomorphism  $\Pi$  from [1] coincides with  $\mathcal{O}_{\psi,\pi}$ .

Thus, by [6, Corollary 4.3] it suffices to show that

$$(\psi, \pi, \mu): (X, A, C_0(G)) \rightarrow (M_B(Y), M(B))$$

is covariant for  $(\sigma, \text{id}_A \otimes 1)$ , in the sense of [6, Definition 2.9]. Thus we must show that

- (i)  $(\pi, \mu)$  is covariant for  $(A, \text{id}_A \otimes 1)$ , and
- (ii)  $\overline{\psi \otimes \text{id}} \circ \sigma(\xi) = \overline{\mu \otimes \text{id}}(w_G) \cdot (\psi(\xi) \otimes 1) \cdot \overline{\mu \otimes \text{id}}(w_G)^*$  for all  $\xi \in X$ .

Condition (i) is immediate because  $\pi$  and  $\mu$  commute. Next, we rewrite (ii) in an equivalent form:

$$(ii)' \quad \overline{\psi \otimes \text{id}}(\sigma(\xi)) \cdot \overline{\mu \otimes \text{id}}(w_G) = \overline{\mu \otimes \text{id}}(w_G) \cdot (\psi(\xi) \otimes 1) \text{ for all } \xi \in X.$$

To proceed further, notice that the maps  $\psi$ ,  $\pi$ , and  $\mu$  from [1] take a particularly simple form in our present context:

- $\psi = \text{id}_X \otimes 1_{M(C_0(G))}$ ;
- $\pi = \text{id}_A \otimes 1_{M(C_0(G))}$ ;
- $\mu = 1_{M(A)} \otimes \text{id}_{C_0(G)}$ .

(We should explain our notation in the above expression for  $\psi$ : it follows from the definitions that, as a Hilbert  $(A \otimes C_0(G))$ -module,  $Y$  coincides with the external tensor product  $X \otimes C_0(G)$  (where  $C_0(G)$  is regarded as a Hilbert module over itself in the canonical way). One just has to keep in mind that  $Y$  does *not* coincide with  $X \otimes C_0(G)$  as a  $B$ -correspondence — the left  $B$ -module multiplication is twisted by the cocycle  $\kappa$ .) Thus, for  $\xi \in X$  we can write:

- $\psi(\xi) = \xi \otimes 1$ ;
- $\overline{\psi \otimes \text{id}}(\sigma(\xi)) = \sigma(\xi)_{13} = \kappa_{13} \cdot (\xi \otimes 1 \otimes 1)$ ;
- $\overline{\mu \otimes \text{id}}(w_G) = 1 \otimes w_G$ .

Since both sides of (ii)' are adjointable Hilbert-module maps from  $B \otimes C^*(G)$  to  $Y \otimes C^*(G)$ , and  $A \odot C_c(G)$  is dense in  $B$ , it suffices to check that the two sides of (ii)' take equal values on elementary tensors of the form  $f \otimes g \otimes a$ , with  $f \in A$ ,  $g \in C_c(G)$ , and  $a \in C^*(G)$ . Evaluating the right-hand side of (ii)' gives

$$\begin{aligned} & ((1 \otimes w_G) \cdot (\xi \otimes 1 \otimes 1)) \cdot (f \otimes g \otimes a) \\ (4.1) \quad &= (1 \otimes w_G) \cdot ((\xi \otimes 1 \otimes 1) \cdot (f \otimes g \otimes a)) \\ &= (1 \otimes w_G) \cdot (\xi \cdot f \otimes g \otimes a). \end{aligned}$$

Now we must use the function-space techniques from Appendix A. We have  $1 \otimes w_G \in C_b(F^0, M^\beta(C^*(G)))$ , with value  $t$  at  $(v, t) \in F^0$ , and

$\xi \cdot f \otimes g \otimes a \in C_c(F^1, C^*(G))$ , so by Corollary A.3 we can evaluate the last quantity in (4.1) at  $(e, t) \in F^1$ , giving

$$\begin{aligned} & (1 \otimes w_G)(r(e, t))(\xi \cdot f \otimes g \otimes a)(e, t) \\ &= (1 \otimes w_G)(r(e), \kappa(e)t)(\xi \cdot f)(e)g(t)a \\ &= \kappa(e)t\xi(e)f(s(e))g(t)a \\ &= \xi(e)f(s(e))g(t)\kappa(e)ta. \end{aligned}$$

We proceed similarly with the left-hand side of (ii)':

$$\begin{aligned} (4.2) \quad & (\kappa_{13} \cdot (\xi \otimes 1 \otimes 1) \cdot (1 \otimes w_G)) \cdot (f \otimes g \otimes a) \\ &= \kappa_{13} \cdot (\xi \cdot f \otimes w_G(g \otimes a)) \end{aligned}$$

Now,  $\kappa_{13} \in C_b(F^1, M^\beta(C^*(G)))$ , with value  $\kappa(e)$  at  $(e, t)$ , and  $\xi \cdot f \otimes w_G(g \otimes a) \in C_c(F^1, C^*(G))$  because  $\xi \cdot f \in C_c(E^1)$  and  $w_G(g \otimes a) \in C_c(G, C^*(G))$ , so by Corollary A.3 we can evaluate the right-hand side of (4.2) at  $(e, t) \in F^1$ , giving

$$\begin{aligned} & \left( \kappa_{13} \cdot (\xi \cdot f \otimes w_G(g \otimes a)) \right)(e, t) \\ &= \kappa_{13}(e, t)(\xi \cdot f \otimes w_G(g \otimes a))(e, t) \\ &= \kappa(e)(\xi \cdot f)(e)(w_G(g \otimes a))(t) \\ &= \kappa(e)\xi(e)f(s(e))w_G(t)(g \otimes a)(t) \\ &= \kappa(e)\xi(e)f(s(e))tg(t)a \\ &= \xi(e)f(s(e))g(t)\kappa(e)ta. \end{aligned}$$

Therefore we have verified (ii)', and this finishes the proof.  $\square$

## APPENDIX A. FUNCTIONS AND MULTIPLIERS

In Section 4 we need to compute with bimodule multipliers in terms of functions. If  $T$  is a locally compact Hausdorff space and  $C$  is a  $C^*$ -algebra, we will use without comment the following identifications (see, e.g., [12] or [3, Appendix C]):

- $C_0(T, C) = C_0(T) \otimes C$ ;
- $M(C_0(T) \otimes C) = C_b(T, M^\beta(C))$ ,

where we write  $M^\beta(C)$  to denote  $M(C)$  with the strict topology. Note that since the action of  $C_b(T, M^\beta(C))$  by multipliers on  $C_0(T, C)$  is via pointwise multiplication, it preserves  $C_c(T, C)$ .

We will need to use functions as multipliers on certain  $C^*$ -correspondences; since this theory is not easily available in the literature, we give details for all the results we need. However, we



make no attempt to construct a general theory — rather, we do only enough to establish Corollary A.3, which we need in Section 4.

For a topological graph  $E$ , we write (as in the rest of the paper)  $A = C_0(E^0)$ ,  $X = X(E)$ , and  $A' = C_0(E^1)$ . We will regard  $X \otimes C$  both as an  $(A' \otimes C) - (A \otimes C)$  correspondence and as an  $(A \otimes C)$ -correspondence.

The following lemma is routine:

**Lemma A.1.**  *$C_c(E^1, C)$  embeds densely in the  $(A' \otimes C) - (A \otimes C)$  correspondence  $X \otimes C$  in the following way: if  $\xi, \eta \in C_c(E^1, C) \subset X \otimes C$ ,  $f \in C_c(E^1, C) \subset A' \otimes C$ , and  $g \in C_c(E^0, C) \subset A \otimes C$ , then  $f \cdot \xi$  and  $\xi \cdot g$  are the elements of  $C_c(E^1, C)$  given by*

$$(A.1) \quad (f \cdot \xi)(e) = f(e)\xi(e)$$

$$(A.2) \quad (\xi \cdot g)(e) = \xi(e) \cdot g(s(e)),$$

and  $\langle \xi, \eta \rangle$  is the element of  $C_c(E^0, C) \subset A \otimes C$  given by

$$(A.3) \quad \langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \xi(e)^* \eta(e).$$

Moreover,  $g \cdot \xi$  is the element of  $C_c(E^1, C)$  given by

$$(A.4) \quad (g \cdot \xi)(e) = g(r(e))\xi(e).$$

*Proof.* First of all, (A.2)–(A.3) make  $C_c(E^1, C)$  into a pre-Hilbert  $C_c(E^0, C)$ -module (where the latter is regarded as a dense  $*$ -subalgebra of  $C_0(E^0, C) = A \otimes C$ ). The only non-obvious property of pre-Hilbert modules is that (A.3) does give an element of  $C_c(E^0, C)$ , but this can be proved by an argument similar to those used in [8, Lemma 1.5].

Observe that the Hilbert-module norm on  $C_c(E^1, C)$  is given by

$$(A.5) \quad \|\xi\| = \sup_{v \in E^0} \left\| \sum_{s(e)=v} \xi(e)^* \xi(e) \right\|^{1/2},$$

which is larger than the uniform norm. In particular, for  $e \in E^1$  the evaluation map  $\xi \mapsto \xi(e)$  from  $C_c(E^1, C)$  to  $C$  is bounded from the Hilbert-module norm to the norm of  $C$ .

Computing with elementary tensors of the form  $\xi \otimes c$  for  $\xi \in C_c(E^1)$  and  $c \in C$ , it is now routine to verify that the completion of the pre-Hilbert module  $C_c(E^1, C)$  is isomorphic to the external tensor product  $X \otimes C$  of the Hilbert  $A$ -module  $X$  and the Hilbert  $C$ -module  $C$ .

Now regarding  $X \otimes C$  as an  $(A' \otimes C) - (A \otimes C)$  correspondence, (A.1) is obviously true on elementary tensors, hence for  $f, \xi \in C_c(E^1) \odot C$ , and therefore as stated by density of  $C_c(E^1) \odot C$  in  $C_c(E^1, C)$  and by continuity of evaluation. Finally, (A.4) follows from (A.1).  $\square$

**Lemma A.2.** *Let  $K \subset E^1$  be compact. On the subspace*

$$C_K(E^1, C) := \{\xi \in C_c(E^1, C) : \text{supp } \xi \subset K\}$$

*of  $X \otimes C$ , the Hilbert-module norm and the uniform norm are equivalent. Consequently,  $C_K(E^1, C)$  is norm-closed in  $X \otimes C$ .*

*Proof.* By (A.5), the uniform norm on  $C_K(E^1, C)$  is smaller than the Hilbert-module norm from  $X \otimes C$ . Thus it suffices to show that the Hilbert-module norm is bounded above by a multiple of the uniform norm. Let  $\xi \in C_K(E^1, C)$ . Using compactness of  $K$  and local homeomorphicity of  $s$ , it is easy to verify that the cardinalities of the intersections  $K \cap s^{-1}(v)$  for  $v \in E^0$  are bounded above by some nonnegative integer  $d$ . Then for any  $v \in E^0$  we have

$$\left\| \sum_{s(e)=v} \xi(e)^* \xi(e) \right\| \leq \sum_{s(e)=v} \|\xi(e)\|^2 \leq d \|\xi\|_u^2,$$

where  $\|\xi\|_u$  denotes the uniform norm of  $\xi$ , and the result follows.  $\square$

Since  $X \otimes C$  is a nondegenerate  $(A' \otimes C) - (A \otimes C)$  correspondence, the left module action of  $A' \otimes C$  extends canonically to the multiplier algebra  $M(A' \otimes C) = C_b(E^1, M^\beta(C))$  (and similarly for the left module action of  $A \otimes C$ ). The following corollary allows us to compute this extended left module action on generators:

**Corollary A.3.** *If  $m \in C_b(E^1, M^\beta(C))$  and  $\xi \in C_c(E^1, C) \subset X \otimes C$ , then the element  $m \cdot \xi$  of  $X \otimes C$  lies in  $C_c(E^1, C)$ , and*

$$(A.6) \quad (m \cdot \xi)(e) = m(e)\xi(e) \quad \text{for } e \in E^1.$$

*If  $n \in C_b(E^0, M^\beta(C))$  then both  $n \cdot \xi$  and  $\xi \cdot n$  lie in  $C_c(E^1, C)$ , and*

$$(n \cdot \xi)(e) = n(r(e))\xi(e);$$

$$(\xi \cdot n)(e) = \xi(e)n(s(e)).$$

*Proof.* Choose a net  $\{m_i\}$  in  $C_c(E^1, C)$  converging strictly to  $m$  in  $M(C_0(E^1, C))$ . The conclusion holds for each  $m_i \cdot \xi$ , by Lemma A.1. Let  $K = \text{supp } \xi$ , a compact subset of  $E^1$ . Then  $m_i \cdot \xi \in C_K(E^1, C)$  for all  $i$ . Since  $m_i \cdot \xi \rightarrow m \cdot \xi$  in the norm of  $X \otimes C$ , we have  $m \cdot \xi \in C_K(E^1, C)$ , by Lemma A.2.

For each  $e \in E^1$ , by norm-continuity of evaluation on  $C_c(E^1, C)$  we have

$$(m \cdot \xi)(e) = \lim_i (m_i \cdot \xi)(e) = \lim_i m_i(e)\xi(e).$$

Moreover, for any  $a \in C$ , we can choose  $f \in C_0(E^1, C)$  such that  $f(e) = a$  and compute:

$$m_i(e)a = m_i(e)f(e) = (m_i f)(e) \rightarrow (mf)(e) = m(e)a.$$

Thus evaluation is strictly continuous on  $C_b(E^1, M^\beta(C))$ ; in particular,

$$\lim_i m_i(e)\xi(e) = m(e)\xi(e),$$

which establishes (A.6).

The statement for  $n \cdot \xi$  follows by composing with the range map  $r: E^1 \rightarrow E^0$ , and the statement for  $\xi \cdot n$  is proved similarly to the above argument for  $m \cdot \xi$ .  $\square$

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